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Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let l =the entire length of the chain, a =the radius of the wheel, x =the longest part hanging down, $l-(\pi a+x)$ =the shortest part, θ =the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right, T and T' the tensions at any points in those parts of the string to which x and $l-(\pi a+x)$ belong, and μ =the coefficient of friction.

We now have, from the usual theory, a unit of length of the chain being taken as a unit of weight.

$$T = Ce^{\mu\theta} + \frac{a}{1+\mu^2}[2\mu\sin\theta + (1-\mu^2)\cos\theta] \dots (1).$$

When $\theta=\frac{1}{2}\pi$, $T=x$, and (1) is

$$x = Ca^{\frac{1}{2}(\mu\pi)} + \frac{2\mu a}{1+\mu^2}, \text{ or } C = \left(x - \frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)} \dots (2),$$

$$\text{and (1) is } T = \left(x - \frac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta-\frac{1}{2}\pi)} + \frac{a}{1+\mu^2}[-2\mu\sin\theta + (1-\mu^2)\cos\theta] \dots (3).$$

In like manner, $T'=l-(\pi a+x)$ when $\theta=-\frac{1}{2}\pi$, and

$$T' = \left(l-\pi a-x + \frac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta+\frac{1}{2}\pi)} + \frac{a}{1+\mu^2}[-2\mu\sin\theta + (1-\mu^2)\cos\theta] \dots (4).$$

For equilibrium, $T=T'$ at the vertex, where $\theta=0$.

$$\therefore \left(l-\pi a-x + \frac{2\mu a}{1+\mu^2}\right)e^{\frac{1}{2}(\mu\pi)} = \left(x - \frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)} \dots (5),$$

$$\text{giving } x = \frac{l-\pi a}{1+e^{-\mu\pi}} + \frac{2\mu a}{1+\mu^2} \dots (6).$$

Then $2x+\pi a-l$ is found, the required length.

Also solved by G. B. M. ZERR.

AVERAGE AND PROBABILITY.

101. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

By direct calculation obtain the average distance between two points in the surface of a circle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P, Q be the random points in the sector AOB . $AO=r$, $OP=y$, $OQ=x$,

$\angle AOB = \beta$, $\angle AOQ = \theta$, $\angle AOP = \varphi$. An element of the sector at P is $ydyd\varphi$; at Q , $xdxd\theta$. The limits of x , are 0 and r ; of y , 0 and x ; of θ , 0 and β ; of φ , 0 and θ .

$$PQ = u = [x^2 + y^2 - 2xy\cos(\theta - \varphi)]^{\frac{1}{2}}.$$

$$\begin{aligned} \therefore \Delta &= \frac{\int_0^\beta \int_0^\theta \int_0^r \int_0^x u x dxdy dy d\theta d\varphi}{\int_0^\beta \int_0^\theta \int_0^r \int_0^x x dxdy d\theta d\varphi} \\ &= \frac{16}{\beta^2 r^4} \int_0^\beta \int_0^\theta \int_0^r \int_0^x u x y d\theta d\varphi dx dy = \frac{8}{3\beta^2 r^4} \int_0^\beta \int_0^\theta \int_0^r \{16\sin^3 \frac{1}{2}(\theta - \varphi) \\ &\quad + 12\sin^3 \frac{1}{2}(\theta - \varphi)\cos(\theta - \varphi) - 2 + 3\cos^2(\theta - \varphi) \\ &\quad + 3\sin^2(\theta - \varphi)\cos(\theta - \varphi)\log[1 + \operatorname{cosec} \frac{1}{2}(\theta - \varphi)]\} x^4 d\theta d\varphi dx \\ &= \frac{8r}{15\beta^2} \int_0^\beta \int_0^\theta \{16\sin^3 \frac{1}{2}(\theta - \varphi) + 12\sin^3 \frac{1}{2}(\theta - \varphi)\cos(\theta - \varphi) - 2 + 3\cos^2(\theta - \varphi) \\ &\quad + 4\sin^2(\theta - \varphi)\cos(\theta - \varphi)\log[1 + \operatorname{cosec} \frac{1}{2}(\theta - \varphi)]\} d\theta d\varphi \\ &= \frac{4r}{45\beta^2} \int_0^\beta \{48\sin^4 \frac{1}{2}\theta \cos \frac{1}{2}\theta + 12\sin^3 \frac{1}{2}\theta \cos \frac{1}{2}\theta - 32\sin^2 \frac{1}{2}\theta \cos \frac{1}{2}\theta - 6\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \\ &\quad - 64\cos \frac{1}{2}\theta + 64 + 9\sin \theta \cos \theta + 6\sin^3 \theta \log(1 + \operatorname{cosec} \theta)\} d\theta \\ &= \frac{2r}{135\beta^2} \{16\sin^6 \frac{1}{2}\beta + 16\cos^6 \frac{1}{2}\beta + 96\sin^5 \frac{1}{2}\beta + 24\sin^4 \frac{1}{2}\beta - 12\cos^4 \frac{1}{2}\beta - 112\sin^3 \frac{1}{2}\beta \\ &\quad - 36\sin^2 \frac{1}{2}\beta - 720\sin \frac{1}{2}\beta + 27\sin^2 \beta + 384\beta - 4 \\ &\quad - 12(\sin^2 \beta \cos \beta + 2\cos \beta + 2)\log(1 + \sin \frac{1}{2}\beta) + 12(\sin^2 \beta \cos \beta \\ &\quad + 2\cos \beta - 2)\log \sin \frac{1}{2}\beta\}. \end{aligned}$$

$$\text{For the circle, } \beta = 2\pi, \Delta = \frac{128r}{45\pi}.$$

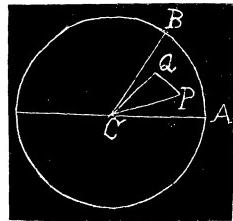
$$\text{For the semicircle, } \beta = \pi, \Delta = \frac{256r}{45\pi} - \frac{1472r}{135\pi^2}.$$

$$\text{For the quadrant, } \beta = \frac{1}{2}\pi.$$

$$\Delta = \frac{32r}{135\pi} \left(48 + \frac{3}{\pi} - \frac{94\sqrt{2}}{\pi} - \frac{6}{\pi} \log \frac{1+\sqrt{2}}{2} \right).$$

II. Solution by the PROPOSER.

Take the center, C , of the circle and the horizontal radius CA as pole and initial line in polar coördinates.



Let $P(x, \theta)$ and $Q(y, \phi)$ be the random points. Then we have

$$PQ = \sqrt{x^2 + y^2 - 2xy\cos(\theta - \phi)}.$$

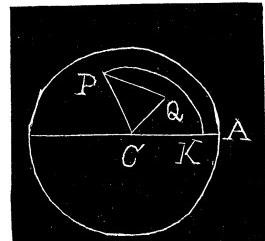
Now with C as a center and a radius CP describe the arc PK .

It is evidently necessary to consider only those positions of the two points in which Q is confined to the sector PCK , since under this limitation PQ will take all possible distances.

An element of area of the circle at the point P is $xdx d\theta$, and at the point Q it is $ydy d\phi$.

The limits of y are 0 and x ; of x , 0 and r ; of ϕ , 0 and θ ; and those of θ , 0 and 2π .

Consequently, we have for the required mean distance



$$\begin{aligned} M &= \frac{1}{(\pi r^2)^2} \int_0^{2\pi} \int_0^\theta \int_0^r \int_0^x \sqrt{x^2 + y^2 - 2xy\cos(\theta - \phi)} x dx d\theta y dy d\phi \\ &= \frac{4}{3\pi^2 r^4} \int_0^{2\pi} \int_0^\theta \int_0^x \left[8\sin^3 \frac{1}{2}(\theta - \phi) + 6\sin^3 \frac{1}{2}(\theta - \phi)\cos(\theta - \phi) \right. \\ &\quad \left. - \frac{3}{2}\sin^2(\theta - \phi)\cos(\theta - \phi) \log\left(\frac{1 + \sin \frac{1}{2}(\theta - \phi)}{\sin \frac{1}{2}(\theta - \phi)}\right) - 1 + \cos^2(\theta - \phi) \right] x^4 d\theta d\phi dx \\ &= \frac{r}{15\pi^2} \int_0^{2\pi} \int_0^\theta \left[8\sin \frac{1}{2}(\theta - \phi) + 40\sin \frac{1}{2}(\theta - \phi)\cos^2 \frac{1}{2}(\theta - \phi) \right. \\ &\quad \left. + 6\sin^2(\theta - \phi)\cos(\theta - \phi) \log\left(\frac{1 + \sin \frac{1}{2}(\theta - \phi)}{\sin \frac{1}{2}(\theta - \phi)}\right) \right. \\ &\quad \left. - 48\sin \frac{1}{2}(\theta - \phi)\cos^4 \frac{1}{2}(\theta - \phi) + 3\cos 2(\theta - \phi) - 1 \right] d\theta d\phi \\ &= \frac{2r}{45\pi^2} \int_0^{2\pi} \left[32 - 32\cos \frac{1}{2}\theta - 16\sin^2 \frac{1}{2}\theta \cos \frac{1}{2}\theta + 24\sin^4 \frac{1}{2}\theta \cos \frac{1}{2}\theta + 3\sin \theta \cos \theta \right. \\ &\quad \left. + 3\sin^3 \theta \log\left(\frac{1 + \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}\right) \right] d\theta = \frac{128r}{45\pi}. \end{aligned}$$

Professor Walker also furnished a second very excellent solution.

MISCELLANEOUS.

91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry" (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Mensuration" (London and Cambridge, 1868), in a table on page 113.